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1987 J. Phys. A: Math. Gen. 20 3301

(http://iopscience.iop.org/0305-4470/20/11/033)

J. Phys. A: Math. Gen. 20 (1987) 3301-3307. Printed in the UK

Electric field induced rotation of spheres

I Turcu
Institute of Isotopic and Molecular Technology, 3400 Cluj-Napoca 5, PO Box 700, Romania

Received 17 July 1986, in final form 24 November 1986

Abstract. A theoretical investigation has been made on the field induced rotation of a sphere immersed in a dielectric liquid and subjected to electric fields at various intensities and frequencies. The regions, in a bidimensional parameter space, with qualitatively different dynamics and the explicit expression of the angular frequency of the sphere were found.

1. Introduction

The spinning of small solid spherical particles when immersed in dielectric liquids and subjected to strong electrostatic fields was observed as early as 1896 by Quincke (1896). The phenomenon exhibits a threshold value of the electric field which depends on certain parameters such as the liquid viscosity and the charge relaxation times.

Cell and protoplast rotations have been observed in both alternating (Teixeira-Pinto *et al* 1960, Pohl and Crane 1971, Holzapfel *et al* 1982) and rotating electric fields (Arnold and Zimmermann 1982). The interpretation of the cell rotation in a rotating field led to a theoretical analysis of the effect in terms of the cell and medium electric properties (Arnold and Zimmermann 1982, Lovelace *et al* 1984).

The occurrence of cell spinning in an alternating electric field was attributed to a dipole-dipole interaction between neighbouring cells (Holzapfel *et al* 1982) which leads to a time-averaged electric torque. The dipole-dipole interaction mechanism is, however, unable to describe the rotation of a single cell far away from the electrodes and from other cells, reported by Pohl and Crane (1971). A second delicate point is the claimed threshold value of the electric field (Mischel and Lamprecht 1980), in contradiction with the theory.

In order to bring some new insight to the problem a theoretical approach on a very simple model is performed. This paper presents the necessary conditions for the appearance of the rotation of a uniform sphere suspended in a dielectric liquid and subjected to a uniform alternating electric field.

2. Quincke rotation of spheres

If a conducting dielectric sphere is immersed in a conducting dielectric liquid and subjected to a uniform electrostatic field it would rotate spontaneously when the field exceeds a threshold value and the following relation is satisfied:

$$\tau_1 < \tau_2 \qquad (1)$$

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where $\tau_1 = \varepsilon_1/\sigma_1$; $\tau_2 = \varepsilon_2/\sigma_2$; ε_1 and ε_2 are the permittivities, σ_1 and σ_2 are the electrical conductivities of the medium and the sphere, respectively, and τ_1 and τ_2 are the charge relaxation times. The phenomenon was reported by Quincke in 1896 and Lampa (1906) was the first to state correctly the general condition (1) for spontaneous rotation. A literature review on the spontaneous rotation phenomenon can be found in a paper by Jones (1984).

The rotation is due to the appearance of a non-zero electric torque:

$$\boldsymbol{T}_e = \boldsymbol{P}_{\text{eff}} \times \boldsymbol{E}_0 \tag{2}$$

which tends to accelerate any initial rotation of the sphere. $\boldsymbol{P}_{\text{eff}}$ and $\boldsymbol{E}_0$ stand for the effective dipole moment of the sphere and the external electric field, respectively.

Following Jones' paper we derive the expression of the effective dipole moment of a rotating sphere in a uniform electrostatic field,

$$\boldsymbol{E}_0 = E_0 \boldsymbol{e}_1 \tag{3}$$

using a rotating coordinate system at rest with respect to the sphere. In the rotating frame the expressions for the field and for the effective dipole moment are:

$$\boldsymbol{E}_0 = \text{Re}[E_0(\boldsymbol{e}_1 - \mathrm{i}\boldsymbol{e}_2)\,\mathrm{e}^{-\mathrm{i}\omega t}]$$

$$\boldsymbol{P}_{\text{eff}} = \text{Re}\left(4\pi\varepsilon_1 R^3 \frac{\hat{\varepsilon}_2 - \hat{\varepsilon}_1}{\hat{\varepsilon}_2 + 2\hat{\varepsilon}_1} E_0(\boldsymbol{e}_1 - \mathrm{i}\boldsymbol{e}_2)\,\mathrm{e}^{-\mathrm{i}\omega t}\right) \tag{4}$$

where ω and R are the angular frequency and the radius of the sphere, respectively, and $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ two orthogonal unit vectors in the rotation plane.

The complex permittivities are:

$$\hat{\varepsilon}_1 = \varepsilon_1 - \mathrm{i}\frac{\sigma_1}{\omega} \qquad \hat{\varepsilon}_2 = \varepsilon_2 - \mathrm{i}\frac{\sigma_2}{\omega}. \tag{5}$$

After some straightforward operations the following expression for T_e, in the frame at rest with respect to the field, is obtained:

$$T_e = 9V\varepsilon_1 E_0^2 \frac{\varepsilon_r - \sigma_r}{(\varepsilon_r + 2)(\sigma_r + 2)} \frac{X}{X^2 + 1} \tag{6}$$

where

$$\varepsilon_r = \frac{\varepsilon_2}{\varepsilon_1} \qquad \sigma_r = \frac{\sigma_2}{\sigma_1} \qquad \tau = \frac{\varepsilon_2 + 2\varepsilon_1}{\sigma_2 + 2\sigma_1} \qquad X = \omega\tau$$

and V is the volume of the sphere.

The viscous forces tend to restrain rotation, the torque imparted by the medium being:

$$T_\eta = -6V\eta\omega \tag{7}$$

where η is the dynamic viscosity of the liquid.

We are interested in the steady state solutions of the equation of motion:

$$I\frac{\mathrm{d}\omega}{\mathrm{d}t} = T_e + T_\eta \tag{8}$$

where I is the momentum of inertia of the sphere. In dimensionless terms equation (8) has the following simple form:

$$\frac{dX}{dt} = CX\left(\frac{p}{X^2+1} - 1\right) \tag{9}$$

where

$$C = \frac{6V\eta}{I} \qquad p = \frac{E_0^2}{E_c^2}\,\mathrm{sgn}(\varepsilon_r - \sigma_r) \qquad E_c^2 = \frac{2\eta(\sigma_r+2)^2}{3\varepsilon_1\tau_1|\varepsilon_r - \sigma_r|}. \tag{10}$$

For $p < 1$ equation (9) admits only the trivial steady state solution:

$$X_1 = 0. \tag{11}$$

For $p > 1$ two new steady state solutions appear:

$$X_1 = 0 \qquad X_{2a,2b} = \pm\sqrt{p-1}. \tag{12}$$

The $\pm$ signs refer to the two possible directions of rotation.

The problem admits two qualitatively different dynamics, the dynamical symmetry breaking being controlled by the magnitude of the field. A critical value E_c is defined so that if $E > E_c$ ($p > 1$) any fluctuation destabilises the rest steady state and a stable rotational motion appears (figure 1). The new solutions are symmetrical and consequently the direction of the rotational motion is dictated by chance. It must be emphasised that $p > 0$ is ensured only if $\varepsilon_r > \sigma_r$, otherwise no rotational motion can occur.

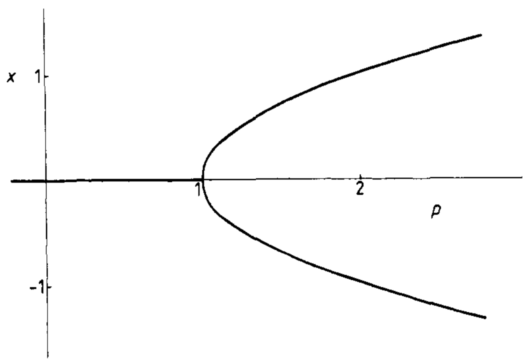

Figure 1. Bifurcation diagram for the Quincke rotational motion.

3. The rotation of spheres in alternating electric fields

An alternating electric field

$$\boldsymbol{E} = E_0\boldsymbol{e}_1 \sin \omega_0 t \tag{13}$$

can be described as the superposition of two rotating fields having opposite angular frequencies. In the rotating coordinate system of the sphere the field and the induced dipole moments have the following expressions:

$$\boldsymbol{E} = \tfrac{1}{2}(\boldsymbol{E}_+ + \boldsymbol{E}_-) \qquad \boldsymbol{P}_{\mathrm{eff}} = \tfrac{1}{2}(\boldsymbol{P}_+ + \boldsymbol{P}_-) \tag{14}$$

where

$$E_\pm = \mathrm{Re}[E_0(e_1 - \mathrm{i}e_2)\,\mathrm{e}^{-\mathrm{i}(\omega\mp\omega_0)t}]$$
$$P_\pm = \mathrm{Re}\left(4\pi\varepsilon_1 R^3 \frac{\hat\varepsilon_2^\pm - \hat\varepsilon_1^\pm}{\hat\varepsilon_2^\pm + 2\hat\varepsilon_1^\pm} E_0(e_1 - \mathrm{i}e_2)\,\mathrm{e}^{-\mathrm{i}(\omega\mp\omega_0)t}\right) \tag{15}$$
$$\hat\varepsilon_1^\pm = \varepsilon_1 - \mathrm{i}\frac{\sigma_1}{\omega\mp\omega_0} \qquad \text{and} \qquad \hat\varepsilon_2^\pm = \varepsilon_2 - \mathrm{i}\frac{\sigma_2}{\omega\mp\omega_0}$$

ω_0 being the angular frequency of the external alternating electric field. The electric torque has four components:

$$T_e = T_{++} + T_{+-} + T_{-+} + T_{--}$$
$$T_{\alpha\beta} = \tfrac{1}{4}(P_\alpha \times E_\beta) \qquad \alpha;\beta = \pm. \tag{16}$$

The second and third terms in equation (16) are time dependent. If $\omega \ll \omega_0$, they are also fast oscillating so that their mean value over a whole period is zero and their contribution can be neglected. In this approximation, in the frame at rest with respect to the fluid, the electric torque has the following expression:

$$T_e = \tfrac{9}{4} V\varepsilon_1 E_0^2 \frac{(\varepsilon_r - \sigma_r)}{(\varepsilon_r + 2)(\sigma_r + 2)}\left(\frac{X + X_0}{(X+X_0)^2+1} + \frac{X - X_0}{(X-X_0)^2+1}\right) \tag{17}$$

with $\omega_0\tau = X_0$.

We are once again interested in the steady state solutions of the equation of motion which, in the alternating electric field case, becomes:

$$\frac{\mathrm{d}X}{\mathrm{d}t} = CX\left(\frac{p}{2}\frac{X^2 - X_0^2 + 1}{[(X+X_0)^2+1][(X-X_0)^2+1]} - 1\right). \tag{18}$$

In addition to the field magnitude, the angular frequency of the external electric field is a new relevant parameter that must be considered. In order to characterise the steady state solutions and their stability, a bidimensional parameter space having the dimensionless parameters p and X_0 as coordinates is introduced.

Besides the trivial solution, equation (18) admits two more pairs of symmetrical steady state solutions:

$$X_1 = 0$$
$$X_{2a,2b} = \pm\tfrac{1}{2}[p + 4(X_0^2 - 1) + (p^2 - 64X_0^2)^{1/2}]^{1/2} \tag{19}$$
$$X_{3a,3b} = \pm\tfrac{1}{2}[p + 4(X_0^2 - 1) - (p^2 - 64X_0^2)^{1/2}]^{1/2}.$$

It should be noted that, if X_1 is defined everywhere in the parameter space, the pairs X_2 and X_3 have physical meaning only in some restricted domains.

Let D_1, D_2, D_3 be defined by:

$$D_1 = \{(X_0, p) | p(1 - X_0^2) < 2(X_0^2 + 1)^2\}$$
$$D_2^+ = \left\{(X_0,p)\,\middle|\, \begin{array}{lll} p(1-X_0^2) > 2(X_0^2+1)^2 & \text{if} & X_0 < \sqrt{2}-1 \\ p > 8X_0 & \text{if} & X_0 > \sqrt{2}-1 \end{array}\right\}$$
$$D_2^- = \left\{(X_0,p)\,\middle|\, \begin{array}{lll} p(1-X_0^2) > 2(X_0^2+1) & \text{if} & 1 < X_0 < \sqrt{2}+1 \\ p < -8X_0 & \text{if} & X_0 > \sqrt{2}+1 \end{array}\right\} \tag{20}$$
$$D_2 = D_2^+ \cup D_2^- \qquad D_3 = D_1 \cap D_2.$$

It is not difficult to verify that X_2 and X_3 are defined only in the domains D_2 and D_3 respectively.

The steady state local stability is easily established by linearising equation (18) in the neighbourhood of the solutions (19). The linearised equations are:

$$\frac{d\delta_1}{dt} = -C\left(1 - \frac{p(1-X_0^2)}{2(X_0^2+1)^2}\right)\delta_1$$

$$\frac{d\delta_{2,3}}{dt} = \mp\frac{C}{2}\frac{[p+4(X_0^2-1)\pm(p^2-64X_0^2)^{1/2}](p^2-64X_0^2)^{1/2}}{p[p\pm(p^2-64X_0^2)^{1/2}]}\delta_{2,3} \tag{21}$$

where

$$\delta_1 = X - X_1 \qquad \delta_2 = X - X_{2a,2b} \qquad \delta_3 = X - X_{3a,3b}. \tag{22}$$

The solution X_1 is locally stable in D_1 and unstable outside it; X_2 is locally stable in D_2 and X_3 is unstable over D_3.

The dynamical behaviour of the system is much more complex than in the case of the electrostatic field. There are some regions in the parameter space with qualitatively different dynamics and with non-zero intersections (figure 2).

It is useful to introduce a generalised potential $U(X)$ defined by the relation:

$$-\frac{\partial U(X)}{\partial X} = CX\left(\frac{p}{2}\frac{X^2-X_0^2+1}{[(X+X_0)^2+1][(X-X_0)^2+1]} - 1\right). \tag{23}$$

The steady state solutions (19) are also the points of extremum for $U(X)$:

$$U(X) = \frac{C}{2}\left(X^2 - \frac{p}{4}\ln[(X+X_0)^2+1][(X-X_0)^2+1]\right). \tag{24}$$

The potential presents three qualitatively different shapes for various values of the parameters X_0 and p (figure 3). In $D_1 - D_3$ it has a single minimum, in X_1. In $D_2 - D_3$ it has a maximum, in X_1, and two symmetrical minima, in X_{2a} and X_{2b} respectively. A more complex shape appears in D_3 where there are three minima, in X_1, X_{2a} and X_{2b}, and two maxima, in X_{3a} and X_{3b}. This is the region where the two different dynamics are both locally stable.

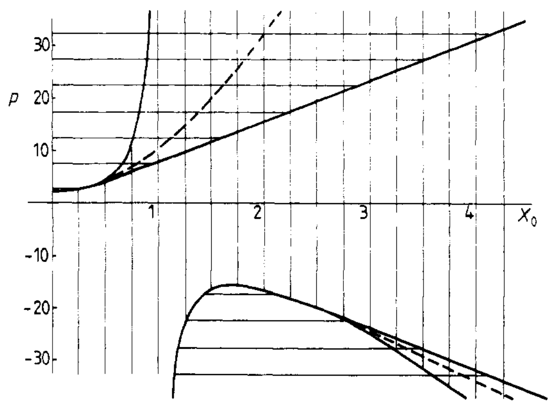

Figure 2. Stability diagram in the bidimensional space of parameters X_0 and p. ≡, region of local stability of the rotational motion; |||, region of local stability of the rest steady state. - - - the curve defined by $F(X_0, p) = 0$.

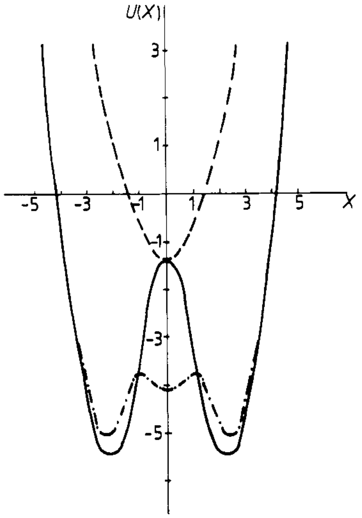

Figure 3. The three possible shapes for the generalised potential $U(X)$: ——, ($X_0 = 0.5$; $p = 12$); — · —, ($X_0 = 1$; $p = 12$); - - -, ($X_0 = 1$; $p = 4$).

The domain D_3 can be divided into two regions by the curve defined by the implicit relation

$$U(X_2) - U(X_1) = (C/8)F(X_0 p) = 0 \tag{25}$$

where

$$F(X_0 p) = p + 4(X_0^2 - 1) + (p^2 - 64X_0^2)^{1/2} - p \ln \frac{p[p + (p^2 - 64X_0^2)^{1/2}]}{8(X_0^2 + 1)}. \tag{26}$$

This curve separates the region with $U(X_1) > U(X_2)$, where X_1 is 'metastable' and X_2 is stable, from the region with $U(X_1) < U(X_2)$ where X_1 is stable and X_2 is 'metastable' (figure 2).

The term 'metastable' is used to designate locally stable steady states which are relatively high minima of the potential.

Consequently, in the domain D_3 the steady states have a 'metastable' behaviour before they disappear or become unstable.

4. Conclusions

The rotation of spheres in the uniform electrostatic field has been extended in the case of the alternating electric field. Using a rotating coordinate system at rest with respect to the sphere we have derived the expression of the electric torque and the equation of motion of the sphere in a uniform alternating electric field.

Sauer and Schlögl (1985), in their extensive and elaborate paper about torques exerted on cylinders and spheres by electromagnetic fields, found no torque on a single sphere in a constant homogeneous field. They are right if the single sphere is at rest but if it rotates with respect to the field a non-zero electric torque appears which is able to increase or to decrease the angular frequency in order to attain a steady state.

The appearance of the spontaneous rotation of spheres can be explained at a phenomenological level by considering the build-up of charge at the interface between the particle and the liquid. When the charge relaxation time of the sphere is lower than that of the liquid, the electric dipole moment produced by the interface charges has the same orientation as the external field and the configuration is stable. If the dipole moment is displaced by rotation, the electric torque tends to bring it back parallel to the field. On the other hand, if the inequality is inverted, the dipole moment itself is inverted and any small rotational displacement produces a torque which tends to further increase the displacement; and the rotational motion can be initiated.

The problem has two relevant parameters that can induce the symmetry breaking: the angular frequency and the magnitude of the external electric field. We have introduced a bidimensional parameter space and we have found the regions where the rotational motion can occur.

By linearising the equation of motion around the steady state solutions, we have found the domains of local stability for these solutions. There are two ways the rotational motion can occur. The first is the usual mechanism: the rest steady state becomes unstable and the rotational motion becomes stable if a critical curve in the parameter space is swept past. The stationary rotational motion develops continuously from the rest state. The second mechanism involves the existence of a middle region where both competing solutions are locally stable. The rest steady state has a 'meta-stable' behaviour before becoming unstable. The stationary rotational motion develops discontinuously by a non-stationary regime. A similar behaviour has the rotational steady state solution when we sweep past the limits of its domain of stability.

References

Arnold W M and Zimmermann V 1982 *Z. Naturf.* **37** 908–15
Holzapfel C, Vienken J and Zimmermann V 1982 *J. Membrane Biol.* **67** 13–26
Jones T B 1984 *IEEE Trans. Ind. Appl.* **IA-20** 845–9
Lampa A S 1906 *Sitzungsberichte dk Akademie der Wissenschaften* **115** 1659–90
Lovelace R V E, Stout D G and Steponkus P L 1984 *J. Membrane Biol.* **82** 157–66
Mischel M and Lamprecht I 1980 *Z. Naturf.* c **35** 1111–3
Pohl H A and Crane J S 1971 *Biophys. J.* **11** 711–27
Quincke G 1896 *Ann. Phys. Chem.* **59** 417–86
Sauer F A and Schlögl R W 1985 *Interaction between electromagnetic fields and cells* (New York: Plenum)
Teixeira-Pinto A A, Nejelski L L, Cutler J L and Heller J H 1960 *Exp. Cell Res.* **20** 548–64